A Look at Saturated Graphs

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Saturation

Definition

Given a graph H, a graph G of order n is said to be ***H*-saturated** provided G contains no copy of H, but the addition of any edge from the complement of G creates a copy of H. That is, given $e \in \bar{G}$, then $G + e$ contains a copy of H.

Extremal vs Saturation Numbers

Definition

The **maximum number** of edges in an H-saturated graph of order n is called the **extremal number (or the Turan number)** for H, and is denoted as $ex(n, H)$.

Definition

The **minimum size** of an H-saturated graph of order n is called the **saturation number** and is denoted as $sat(n, H)$.

Fundamental Questions on Saturated Graphs

1. Given a graph G, what is $ex(n, G)$?

2. Given a graph G, what is $sat(n, G)$?

3. What sizes of G-saturated graphs are possible between the saturation number and extremal number? This set of values is called the **saturation spectrum of G**.

4. When are all values possible?

5. What general theory can we build for $ex(n, G)$ or $sat(n, G)$?

Classic extremal results

Turan (1941) provided the extremal number for complete graphs:

Theorem

Among graphs of order n which do not contain K_t, there exists exactly one with the maximum number of edges, the complete, balanced, $(t-1)$-partite graph.

For triangles, this is the balanced comple bipartite graph, thus $ext(n, K_3) = \lfloor n/2 \rfloor \lceil n/2 \rceil$. (Determined by W. Mantel in 1906.)

The Erdős-Stone Theorem, 1946

Established the magnitude of the extremal number for all graphs with chromatic number at least 3.

Theorem

$$lim_{n \to \infty} \frac{ex(n,G)}{n^2} = \frac{1}{2}\left(1 - \frac{1}{\chi(G)-1}\right).$$

Saturation for Complete Graphs

In 1964 **Erdős, Hajnal and Moon** determined that:

Theorem

$$sat(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}.$$

This arises from the graph $K_{t-2} + \overline{K}_{n-t+2}$, where $+$ denotes join.

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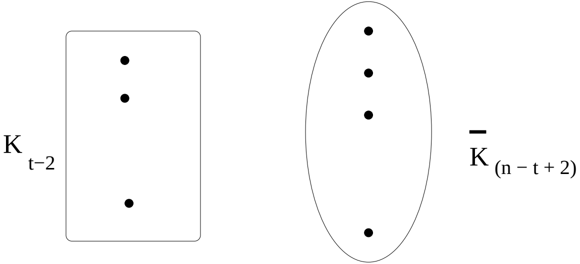

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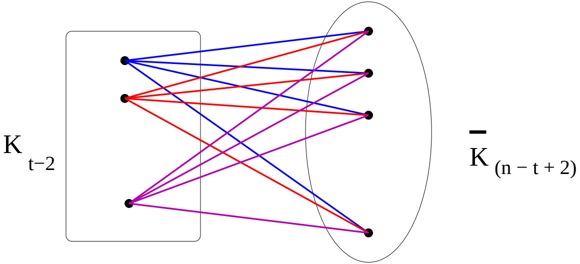


This means that for a triangle, $sat(n, K_3) = n - 1$.

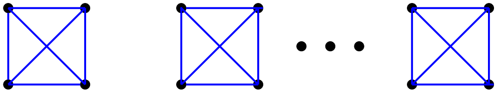
Useful properties of extremal numbers

Let **F** be a family of graphs. Then $ex(n, \mathbf{F})$ satisfies:

1. $ex(n, \mathbf{F}) \leq ex(n+1, \mathbf{F})$.
2. If $\mathbf{F_1} \subset \mathbf{F}$ then $ex(n, \mathbf{F_1}) \leq ex(n, \mathbf{F})$.
3. If $H \subseteq G$, then $ex(n, H) \leq ex(n, G)$.

Problems with saturation numbers

However, these rules do not hold in general for saturation numbers. Example of 3. Consider K_4 and a supergraph H obtained by attaching an additional edge to K_4. We know that $sat(n, K_4) = 2n - 3$. But for H we have:



$$here \;\; n = 4m \;\; and \;\; size = 6m$$

$$hence \;\; size = \; 3n/2$$

Thus, $sat(n, H) \leq 3n/2$.

Slight Aside

Thus we have seen that the extremal number for triangles is $O(n^2)$ while the saturation number is $O(n)$.

This is no accident!

Kászonyi and Tuza provided a very general upper bound on saturation numbers and used it to show the following:

Kászonyi and Tuza, 1986

Theorem

For every graph F there exists a constant c such that

$$sat(n, F) < cn.$$

Back to Cliques: K_3

In 1995 **Barefoot, Casey, Fisher, Fraughnaugh and Harary**, showed the following:

Theorem

For $n \geq 5$, there exists a K_3-saturated graph of order n with m edges if and only if it is complete bipartite or

$$2n - 5 \leq m \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 1.$$

Note: A gap at the bottom, between $n-1$ and $2n-5$. This is a result of a combination of connectivity and the fact that triangle saturated graphs have diameter two. It is then easy to show the gap at the bottom exists. At the top, extremal theory and convexity suffice.

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Theorem

(*Barefoot, et al*) *Every 2-connected graph of order n and diameter two has at least $2n - 5$ edges.*

Note

This result works well for triangles, $K_4 - e$ and other graphs where the saturation graph must have diameter 2.

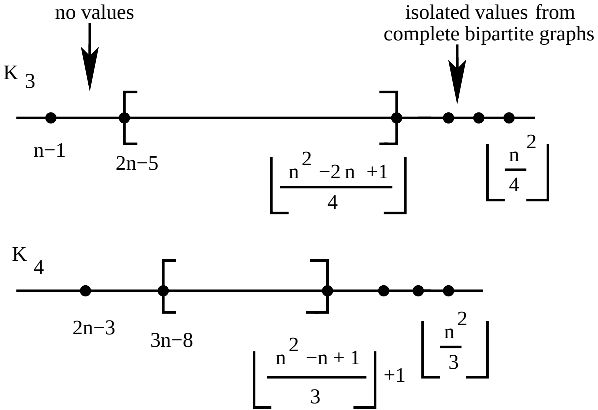
With **K. Amin, J. Faudree and E. Sidorowicz** (2013) we were able to generalize this result for all $t \geq 3$.

Theorem

For $n \geq 3t + 4$ and $t \geq 3$, there is a K_t-saturated graph G of order n with m edges if, and only if, G is complete $(t-1)$-partite or

$$(t-1)(n-t/2) - 2 \leq m \leq \lfloor \frac{(t-2)n^2 - 2n + (t-2)}{2(t-1)} \rfloor + 1.$$

Note, same sort of gaps exist. Also note this reduces to the Barefoot et al result when $t = 3$.


no values
isolated values from
complete bipartite graphs
K 3
n−1
2n−5
$\left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$
$\left\lfloor \frac{n^2}{4} \right\rfloor$
K 4
2n−3
3n−8
$\left\lfloor \frac{n^2-n+1}{3} \right\rfloor +1$
$\left\lfloor \frac{n^2}{3} \right\rfloor$

Saturation number for paths

Kászonyi and Tuza, 1986:

Theorem

1. *For* $n \geq 3$, $sat(n, P_3) = \lfloor n/2 \rfloor$.
2. *For* $n \geq 4$,

$$sat(n, P_4) = \begin{cases} n/2 & n \text{ even} \\ (n+3)/2 & n \text{ odd.} \end{cases}$$

3. *For* $n \geq 5$, $sat(n, P_5) = \lceil \frac{5n-4}{6} \rceil$.
4. *Let*

$$a_k = \begin{cases} 3 \cdot 2^{t-1} - 2 & \text{if} \quad k = 2t \\ 4 \cdot 2^{t-1} - 2 & \text{if} \quad k = 2t+1. \end{cases}$$

then if $n \geq a_k$ *and* $k \geq 6$, $sat(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$.

ext(n, P_k) - by several including Faudree and Schelp

Theorem

For all $n \geq 3$,

1.

$$ext(n, P_4) = \begin{cases} n \ if \ n \equiv 0(mod3) \\ n-1 \ if \ n \equiv 1,2(mod3). \end{cases}$$

2.

$$ext(n, P_5) = \begin{cases} 3n/2 \ if \ n \equiv 0(mod4) \\ 3n/2-2, \ if \ n \equiv 2(mod4) \\ 3(n-1)/2, \ if \ n \equiv 1,3(mod4). \end{cases}$$

3.

$$ext(n, P_6) = \begin{cases} 2n, \ if \ n \equiv 1(mod5) \\ 2n-2, \ if \ n \equiv 1,4(mod5) \\ 2n-3, \ if \ n \equiv 2,3(mod5). \end{cases}$$

Small Path Sat Spectrum

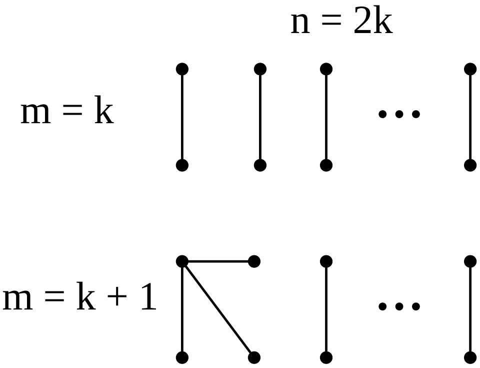
Work with **W. Tang, E. Wei, C.Q. Zhang** (2012).
If we consider P_3 it is simple to see that:

Theorem

$$sat(n, P_3) = ex(n, P_3) = \lfloor n/2 \rfloor.$$

There is a simple procedure for evolving a P_4-saturated graph from the saturation number to the extremal number, one edge at a time. Thus, the spectrum for P_4 is complete.

P_4 has a continuous saturation spectrum



etc.

Do any other graphs have a continuous spectrum?

With J. Faudree, R. Faudree, M. Jacobson and B. Thomas (2009):

Theorem

If $t \geq 3$ and $n \geq t+1$, then the spectrum of the star $K_{1,t}$ is continuous from $sat(n, K_{1,t})$ to $ex(n, K_{1,t})$.

For completeness

Kászonyi and Tuza:

Theorem

$$sat(n, K_{1,t}) = \begin{cases} \binom{t}{2} + \binom{n-t}{2} & \text{if } t+1 \leq n \leq t + t/2 \\ \lceil \frac{t-1}{2} n \rceil - t^2/8 & \text{if } t + 1/2 \leq n. \end{cases}$$

Folklore??? Obvious!

Theorem

$ex(n, K_{1,t}) = \lfloor \frac{t-1}{2} n \rfloor$. *That is, a graph that is* $t-1$*-regular or nearly regular.*

P_5 with Tang, Wei and Zhang

Here things get a little bit more complicated.

Theorem

Let $n \geq 5$ and $sat(n, P_5) \leq m \leq ext(n, P_5)$ be integers. Then there exists an (n, m) P_5-saturated graph if and only if $n \equiv 1, 2(mod 4)$, or

$$m \neq \begin{cases} \frac{3n-5}{2} \text{ if } n \equiv 3(mod 4) \\ \frac{3n}{2} - j, j = 1, 2, \text{ or } 3 \text{ if } n \equiv 0(mod 4). \end{cases}$$

P_6

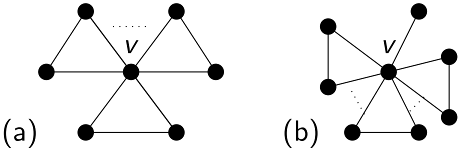
For P_6 there are 5 cases depending on the value of $n(mod 5)$.

There are gaps in the spectrum for each case. We will see more on this later.

Saturation for cliques minus an edge

The extremal number for $K_4 - e$ is achieved by the complete bipartite graph.

$sat(n, K_4 - e) = \lfloor \frac{3(n-1)}{2} \rfloor$, and is achieved by:



Saturation spectrum for $K_4 - e$

With Jessica Fuller we showed:

Theorem

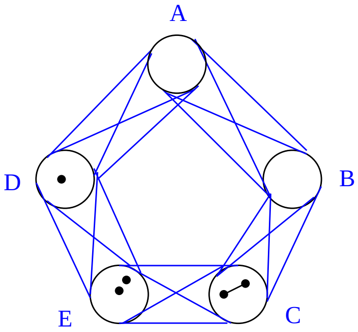
If G is a $K_4 - e$ saturated graph on n vertices, then either G is a complete bipartite graph, a 3-partite graph (like the saturation graph of the previous frame), or has size in the interval

$$[2n - 4, \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 6]$$

Here the gap between the saturation number and 2n-4 happens for reasons similar to that for triangles.

A look at the proof

Case: Suppose $4n - 18 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 5$.
Here $|A| = n - |B| - |C| - 5$, $|B| = b \geq 2$, $|C| = c \geq 2$, $|D| = 2$ and $|E| = 3$.



Then $m = (n - c)(c + 2) - 5c + b - 4$. So as b increases by 1, with c fixed, then m increases by 1.

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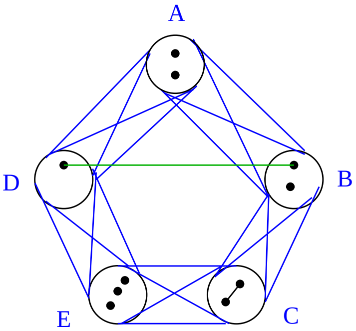

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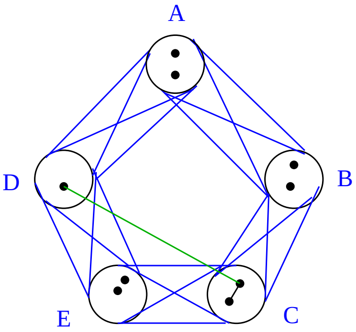

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Larger Cliques Minus an Edge

It is straight-forward to extend the $K_4 - e$ interval to larger cliques:

Theorem

There are $K_t - e$ saturated graphs in the interval
$\left[(t-2)n - \binom{t-1}{2} - 1, \left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t-3)n - \binom{t-2}{2} - 1\right]$.

Also, there are $(K_t - e)$-saturated graphs for sporadic values of m in

$$\left[\left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t-3)n - \binom{t-2}{2} + 4, \left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil\right.$$
$$\left. +(t-2)n - \binom{t-1}{2} - 1.\right]$$

Definition

The fan F_t is the graph consisting of t edge-disjoint triangles that intersect at a single vertex v.

Note: The fan is sometimes called a friendship graph.

Extremal Number for the Fan F_k

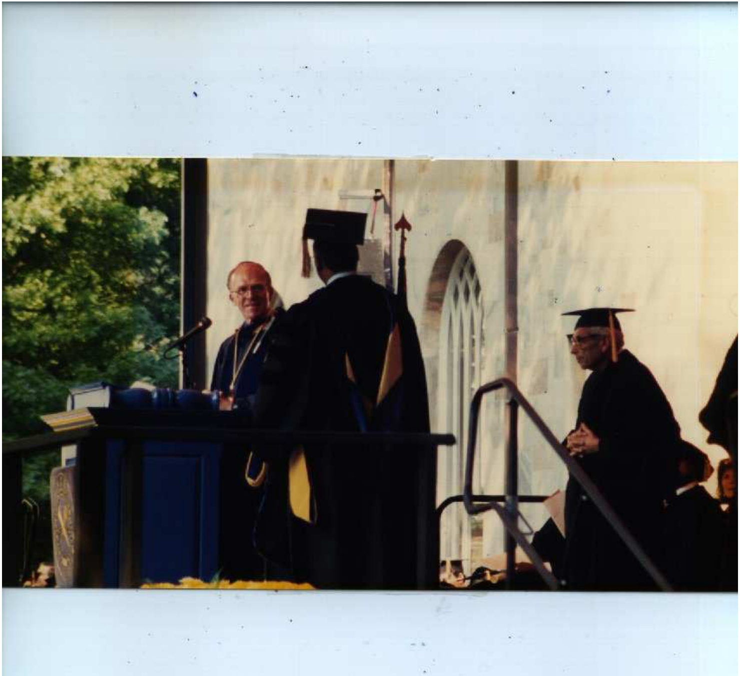
With Erdős, Furedi and Gunderson (1995) we determined the extremal number for fans F_t.

Theorem

For every $t \geq 1$, and for every $n \geq 50t^2$, if a graph G on n vertices has more than

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} t^2 - t & \text{if } t \text{ is odd} \\ t^2 - \frac{3}{2}t & \text{if } t \text{ is even} \end{cases}$$

edges, then G contains a copy of the t-fan, F_t. Furthermore, the number of edges is best possible.



Saturation Spectrum for fans

With J. Fuller we showed the following:

Theorem

For $t \geq 2$, and $n \geq 3t - 1$, $sat(n, F_t) = n + 3t - 4$.

Theorem

There exists an F_2-saturated graph G on $n \geq 7$ vertices and m edges where $m = n + 2$, or $2n - 4 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2$, or m is the size of a complete bipartite graph with one additional edge.

We also showed:

Theorem

There exists an F_3-saturated graph G of order n with m edges for $m = n + 5$, or $2n + 2 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 5$, or m equals the size of a complete bipartite graph with six added edges. Further, the only unknown possible sizes are from $2n - 5$ to $2n + 1$.

F_4 gets even more interesting

Theorem

There exists an F_4-saturated graph G on n vertices and m edges for $m = n + 8$, or $3n + 2 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 10$ except for $4n - 8$, or G is a complete bipartite graph with the proper 10 additional edges.

Saturation for unions of cliques

With R. Faudree, M. Ferrara, and M. Jacobson (2009):
First tK_p.

Theorem

Let $t \geq 1$, $p \geq 3$ and $n \geq p(p+1)t - p^2 + 2p - 6$ be integers. Then

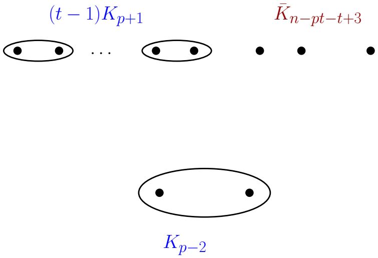
$$sat(tK_p, n) = (t-1)\binom{p+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2).$$

Theorem

Let $2 \leq p \leq q$ and $n \geq q(q+1) + 3(p-2)$ be integers. Then

$$sat(K_p \cup K_q, n) = (p-2)(n-p+2) + \binom{p-2}{2} + \binom{q+1}{2}.$$

Proof Idea



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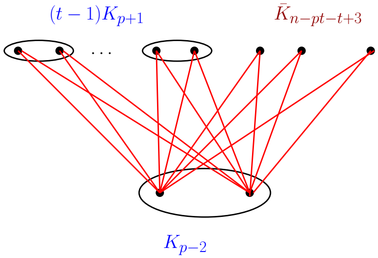

Definition

Let the graph comprised of t copies of K_p intersecting in a common K_ℓ be called a generalized fan and be denoted $F_{p,\ell}$

with Faudree, Ferrara, and Jacobson

Theorem

Let $p \geq 3, t \geq 2$ and $p-2 \geq \ell \geq 1$ be integers. Then, for sufficiently large n,

$$sat(F_{p,\ell}, n) = (p-2)(n-p+2) + \binom{p-2}{2} + (t-1)\binom{p-\ell+1}{2}.$$

Definition

A graph on $(r-1)k+1$ vertices consisting of k cliques each with r vertices, which intersect in exactly one common vertex, is called a $(k,r)-fan$.

Theorem

For every $k \geq 1$, and for every $n \geq 16k^3r^8$, if a graph G on n vertices has more than

$$ex(n, K_r) + \begin{cases} k^2 - k & \text{if } k \text{ is odd} \\ k^2 - \frac{3}{2}k & \text{if } t \text{ is even} \end{cases}$$

edges, then G contains a copy of the (k,r)-fan. Furthermore, the number of edges is best possible.

To see the last result is best possible consider:

For odd k take the Turan graph and embed two vertex disjoint copies of K_k in one partite set.

For even k take the Turan graph and embed a graph with $2k - 1$ vertices and $k^2 - (3/2)k$ edges with max degree $k - 1$ in one partite set.

Gaps: work with P. Horn,, M. Jacobson and B. Thomas

Definition

A tree T of order ℓ, $T \neq K_{1,\ell-1}$, having a vertex that is adjacent to at least $\lfloor \frac{\ell}{2} \rfloor$ leaves is called a scrub-grass tree.

Theorem

Let T be a path or scrub-grass tree on $\ell \geq 6$ vertices and $n = |G| \equiv 0 \ mod(\ell - 1)$ and m be an integer such that $1 \leq m \leq \lfloor \frac{\ell-2}{2} \rfloor - 1$. There is no graph of size $\frac{n}{\ell-1}\binom{l-1}{2} - m$ in the spectrum of T. Hence, there is a gap in the spectrum.

Definition

A graph F is weakly G-saturated if F does not contain a copy of G, but there is an ordering of the missing edges of G so that if they are added one at a time, each edge creates a new copy of F. The minimum size of a weakly F-saturated graph G of order n is denoted $wsat(N, F)$.

Question:

Question

For which graphs G is $sat(n, G) = wsat(n, G)$?

Bollobás (1967) showed the following:

Theorem

$$wsat(n, K_p) = sat(n, K_p).$$

With R. Faudree and M. Jacobson, 2013

If F is a graph of order p and size q:

Theorem

$$\frac{\delta n}{2} - \frac{n}{\delta + 1} \leq wsat(n, F) \leq (\delta - 1)n + (p - 1)\frac{p - 2\delta}{2}.$$

We further showed that for any tree T_p on p vertices:

Theorem

$$p - 2 \leq wsat(n, T_p) \leq \binom{p-1}{2}.$$

More on trees

For a labeled tree T_p

Theorem

$$lim_{n \to \infty} P(wsat(n, T_p)) = p - 2 \to 1.$$

Proof uses Cayley tree formula.

weak sat for multiple copies

with R. Faudree (2014):

Theorem

$$wsat(n, kK_t) = (t-2)n + k - (t^2 - 3t + 4)/2.$$

Theorem

$$wsat(n, kC_t) = \begin{cases} n+k-2 & \text{if } t \text{ is odd} \\ n+k-1 & \text{if } t \text{ is even.} \end{cases}$$